

[This question paper contains 2 printed pages.]

Your Roll No.....

Sr. No. of Question Paper : 4686 E
Unique Paper Code : 32351402
Name of the Paper : Riemann Integration and Series of Functions
Name of the Course : B.Sc. (H) Mathematics
Semester : IV

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **two** parts from each question.

- 1(a) Let f be integrable on $[a, b]$, and suppose g is a function on $[a, b]$ such that $g(x) = f(x)$ except for finitely many x in $[a, b]$. Show g is integrable and $\int_a^b g = \int_a^b f$ (6)
- (b) Show that if f is integrable on $[a, b]$ then f^2 also is integrable on $[a, b]$. (6)
- (c) (i) Let f be a continuous function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$. Show that if $\int_a^b f(x) dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$ (3)
- (ii) Give an example of function such that $|f|$ is integrable on $[0, 1]$ but f is not integrable on $[0, 1]$. Justify it. (3)

2(a) State and prove Fundamental Theorem of Calculus I. (6.5)

(b) State Intermediate Value Theorem for Integrals. Evaluate $\lim_{x \rightarrow 0} \frac{1}{x} \int_0^x e^{t^2} dt$. (6.5)

(c) Let function $f: [0, 1] \rightarrow R$ be defined as

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Calculate the upper and lower Darboux integrals for f on the interval $[0, 1]$. Is f integrable on $[0, 1]$? (6.5)

3(a) Examine the convergence of the improper integral $\int_0^\infty e^{-x} x^{n-1} dx$. (6)

(b) Show that the improper integral $\int_\pi^\infty \frac{\sin x}{x} dx$ is convergent but not absolutely convergent. (6)

P.T.O.

- (c) Determine the convergence or divergence of the improper integral (6)

(i) $\int_0^1 \frac{dx}{x(\ln x)^2}$

(ii) $\int_1^\infty \frac{x dx}{\sqrt{x^3+x}}$

- 4(a) Show that the sequence

$$f_n(x) = \frac{nx}{1+nx}, \quad x \in [0,1], \quad n \in \mathbb{N}$$

converges non-uniformly to an integrable function f on $[0,1]$ such that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 f(x) dx \quad (6.5)$$

- (b) Show that the sequence $\{x^2 e^{-nx}\}$ converges uniformly on $[0, \infty)$. (6.5)

- (c) Let $\langle f_n \rangle$ be a sequence of continuous function on $A \subset \mathbb{R}$ and suppose that $\langle f_n \rangle$ converges uniformly on A to a function $f: A \rightarrow \mathbb{R}$. Show that f is continuous on A . (6.5)

- 5(a) Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $x \geq 0$. Show that sequence $\langle f_n \rangle$ converges non-uniformly on $[0, \infty)$ and converges uniformly on $[a, \infty)$, $a > 0$. (6.5)

- (b) State and prove Weierstrass M test for the uniform Convergence of a series of functions. (6.5)

- (c) Show that the series of functions $\sum \frac{1}{n^2+x^2}$, converges uniformly on \mathbb{R} to a continuous function. (6.5)

- 6(a) (i) Find the exact interval of convergence of the power series (3)

$$\sum_{n=0}^{\infty} 2^{-n} x^{3n}$$

- (ii) Define $\sin x$ as a power series and find its radius of convergence (3)

- (b) Prove that $\sum_{n=1}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}$ for $|x| < 1$ and hence evaluate $\frac{\sum_{n=1}^{\infty} n^2 (-1)^n}{3^n}$. (6)

- (c) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{for } |x| < R. \quad (6)$$